Absorbing Boundary Conditions for the TLM Method

Juan A. Morente, Jorge A. Portí, and Mohsine Khalladi

Abstract—**The numerical behavior of different absorbing boundary conditions when applied to the transmission-line modelling method is presented. These conditions may be classified into three different groups according to the way they are derived. The first group is obtained by discretizing one-way analytical conditions derived for the analytical wave equation. The second group is a set of discrete conditions directly obtained for the discrete wave equation. The last group is based on appropriate reflection coefficients derived purely from transmission-line theory. Due to its different behavior, the numerical study is explicitly carried out for both two- and three-dimensional free-space scattering problems.**

I. Introduction

The Transmission-Line Modelling (TLM) method has been successfully applied during the last twenty years for the solution of electromagnetic-wave-propagation problems. Details, applications and advantages of the method are readily available in the literature [1]–[5].

Since its formulation in 1971 by Johns [1], many of the problems studied using the TLM method have been concerned with waveguide devices due to its simplicity to implement electric and magnetic walls. In order to study free-space problems, absorbing boundary conditions have to be established to avoid non-physical reflections. Although the method has been used to solve a number of free-space problems [6], little has been said about the truncation conditions. It was not until the late 80's that absorbing boundary conditions applied to the TLM method appeared explicitly in the literature [7]–[8]. These conditions are based on the definition of appropriate reflection coefficients derived purely by means of transmission-line considerations and so will be termed match-termination conditions.

Two other types of absorbing boundary conditions are considered in this paper. The first type, which we will refer to as family of one-way equations, is based on discrete approximations of analytical conditions for the analytical wave equation. These conditions have been successfully applied to the finite-difference method, whose equivalence with the TLM method under certain circumstances has been well established [9]. Since the second group of conditions is directly concerned with finding discrete boundary conditions for the discretized wave equation, they will therefore be labelled discrete boundary conditions.

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Most of the previous analytical and numerical works on truncation conditions assume that results for the wave equation of order n are extrapolable for different values of n, so the study is usually limited to two-dimensional problems. A surprising example of how Huyghens' principle only holds for odd values of n is presented in [10]. This, added to the fact that numerical treatment may produce unstabilities that can differ depending on the value of n, justifies the need of an explicit and separate study of two- and three-dimensional scattering problems.

II. Absorbing Boundary Conditions

This section presents discrete approximations of the absorbing boundary conditions for a plane wave travelling in the $-x$ direction (boundary $x = 0$). In the following equations, $U^n(i, l, m)$ stands for the scattered voltage pulses at spatial point $(i\,\Delta x, l\,\Delta y, m\,\Delta z)$ and time $n\,\delta t$. That is to say, the pulses calculated in the absence of the scatterer have been subtracted prior to applying the conditions, for they correspond to the incident fields and are treated separately using symmetry considerations.

Considering the way they are derived, we classify the boundary conditions into the following three groups:

1) One-Way Equations: These are analytical boundary conditions for the analytical wave equation. A detailed description is available in [11]. The condition, in its discrete form, is

$$U^{n+1}(0, l, m) = \frac{1}{p_0 + 1}\Lambda_1 - \frac{p_0}{p_0 + 1}\Lambda_2 - \frac{p_2}{p_0 + 2}(\Lambda_3 + \Lambda_4) \tag{1}$$

where

$$\begin{aligned}
\Lambda_1 &= U^{n+1}(1, l, m) - U^{n-1}(1, l, m) + U^{n-1}(0, l, m)\\
\Lambda_2 &= U^{n+1}(1, l, m) - 2\,U^n(1, l, m) - 2\,U^n(0, l, m)\\
&\quad + U^{n-1}(1, l, m) + U^{n-1}(0, l, m)\\
\Lambda_3 &= U^n(1, l+1, m) - 2\,U^n(1, l, m)\\
&\quad + U^n(1, l-1, m) + U^n(0, l+1, m)\\
&\quad - 2\,U^n(0, l, m) + U^n(0, l-1, m)\\
\Lambda_4 &= U^n(1, l, m+1) - 2\,U^n(1, l, m)\\
&\quad + U^n(1, l, m-1) + U^n(0, l, m+1)\\
&\quad - 2\,U^n(0, l, m) + U^n(0, l, m-1)
\end{aligned} \tag{2}$$

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For two-dimensional problems the term Λ_4 in (1) and (2) vanishes.

Numerical results will be presented for the Padé approximant, $p_0 = 1$ and $p_2 = -0.5$, corresponding to the second-order condition proposed by Engquist and Majda in [12]. The case $p_0 = 1$ and $p_2 = 0$, obtained by deriving with respect to time a first-order condition, is also presented.

2) Discrete Boundary Conditions: They are a set of discrete boundary conditions (first and second order) directly derived for the discrete form of the wave equation [13].

2.1) Space-time extrapolation.

$$U^{n+1}(0, l, m) = U^n(1, l, m)$$
$$U^{n+1}(0, l, m) = 2\, U^n(1, l, m) - U^{n-1}(2, l, m) \quad (3)$$

2.2) Averaging method.

$$U^{n+1}(0, l, m) = \tfrac{1}{3}\{U^{n+1}(1, l, m) + U^n(0, l, m) + U^n(1, l, m)\}$$
$$U^{n+1}(0, l, m) = \tfrac{1}{9}\{6\, U^{n+1}(1, l, m) - U^{n+1}(2, l, m) + 6\, U^n(0, l, m) + 4\, U^n(1, l, m) - 2\, U^n(2, l, m) - U^{n-1}(0, l, m) - 2\, U^{n-1}(1, l, m) - U^{n-1}(2, l, m)\} \quad (4)$$

3) Match Termination of the Mesh: This group, expressly derived for the TLM method, is based on the definition of appropriate reflection and transmission coefficients obtained using transmission-line theory [7]–[8]. The general form of this coefficient appears in [7]. For the case of symmetrical condensed node with identical dimensions in all directions and no series or shunt stubs, the reflection coefficient becomes zero and, thus, the absorbing condition is

$$U^{n+1}(0, l, m) = 0 \quad (5)$$

Third and higher-order conditions can be found for both one-way equation and discrete boundary conditions. However, they are not included in this paper because they have been shown to produce unstabilities due to low-frequency modes, as predicted by Hidgon in [13].

III. Numerical Results

A. Two-Dimensional Problem

The geometry depicted in Fig. 1 is used to test the performances of the different two-dimensional absorbing boundary conditions. A gaussian pulse defined by

$$E_z = E_0 e^{-g^2(t - t_{max})^2} \quad (6)$$

with $E_0 = 1$ V/m, $g = 3$ ns^{-1} and $t_{max} = 0.715$ ns, travelling in the $+x$ direction, illuminates an infinitely-long square cylinder of side 10 Δ, with $\Delta = 0.05$ m and $\delta = \Delta/c = 0.17$ ns. The space domain D_1 has been modelled by means of a square mesh of side 40 Δ and a reference solution is obtained by solving the problem for an extended domain D_e of side 100 Δ. The output signal chosen is E_z in the shadow region (mesh point (59,50) in the figure), where the response takes relatively small values, in order to better discriminate slight differences due to non-physical reflections. The performance of the conditions will be quantitatively stated by evaluating the following error parameter

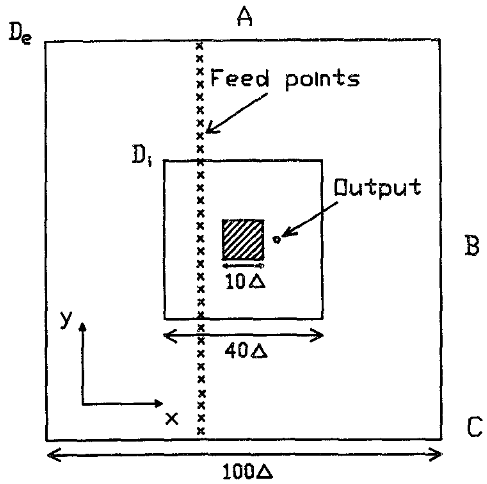

Fig. 1. Geometry of the two-dimensional problem.

$$\text{Error}^n = \sqrt{\sum_{r=1}^{n} (\text{OUT}_i^r - \text{OUT}_e^r)^2} \quad (7)$$

where OUT_i^r and OUT_e^r are the output signals at time step r and evaluated for the reduced and extended domains, respectively.

With the mesh geometry described above, exact values of the reference solution are expected until time step 160 approximately, where reflection from point A in the figure arrives at the output point. Non-physical reflections originating at points B and C are expected at later times of about 230 δ and 290 δ. Reflections from point A are shown to be almost negligible for the reference solution and we will consider as valid values of the error parameter for times below time step 230.

Fig. 2 compares the absorbing boundary conditions presented above. Similar behavior is observed for the second-order Padé approximant and match termination but substantial reduction of reflections is achieved using discrete boundary conditions. The same reduction is observed for the one-way condition with $p_0 = 1$ and $p_2 = 0$ (one-way 1 in the figure) that corresponds to a first-order condition when derived with respect to time. It may also be observed that better performance is achieved for second-order conditions than for first-order ones. It has been found that less error-parameter values are initially obtained for third-order conditions but, as mentioned above, low frequencies cause unstabilities to appear at later times.

Fig. 3 illustrates the reference output field, the solution obtained for the second-order space-time extrapolation, and the solution for the match-termination conditions for comparison. Fig. 4 shows the output field evaluated for

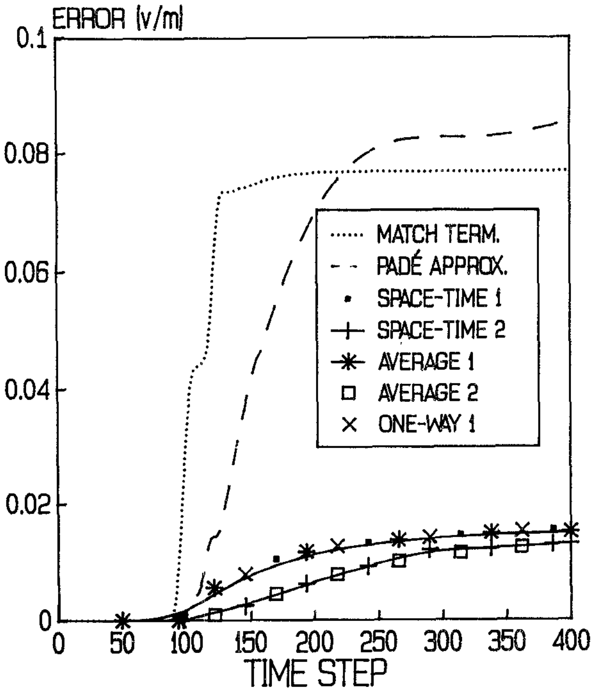

Fig. 2. Comparison of absorbing boundary conditions for the two-dimensional problem.

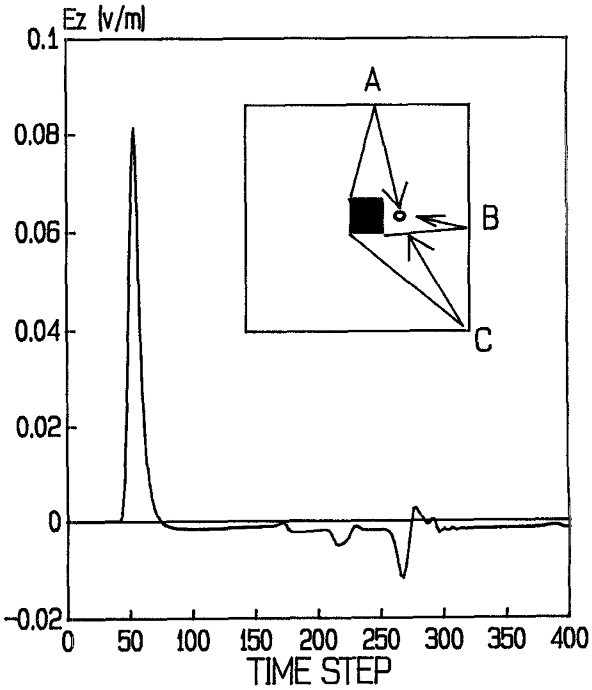

Fig. 4. Electric field at the output point of Fig. 1 for the match-termination condition using the extended domain D_e.

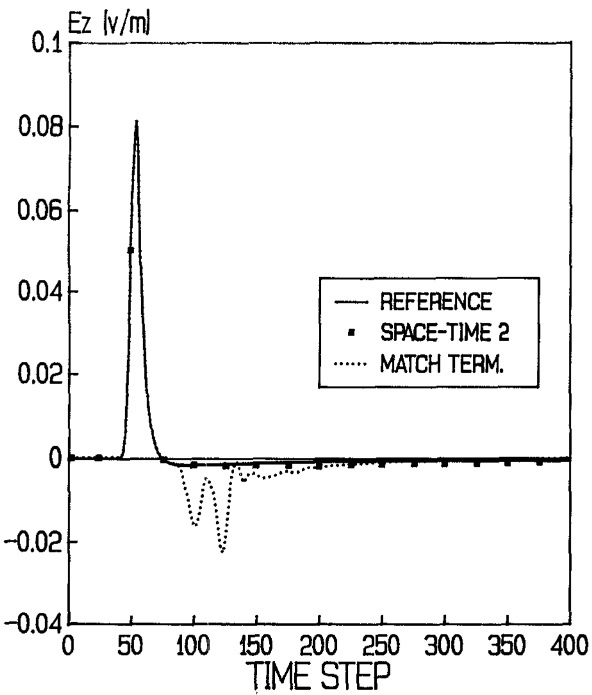

Fig. 3. Electric field at the output point of Fig. 1 for the reference solution and two absorbing boundary conditions.

the extended domain D_e using match-termination conditions. It is evident from these figures that match termination is not capable of efficiently absorbing outgoing waves, producing instead non-physical reflections that, according to the time they appear, have their origin at points A, B or C of Fig. 1. For second-order Padé approximation of one-way conditions, reflection from point B in the figure does not appear because this approximant produces exact absorption for normal incidence.

B. Three Dimensional Problem

A similar geometry is used to test boundary conditions in three-dimensional problems except that the gaussian pulse incides upon a cube of side 10 Δ. The problem is solved in a reduced domain D_i of dimensions 40 Δ $\times$ 40 Δ $\times$ 40 Δ. An extended domain D_e, 100 Δ $\times$ 100 Δ $\times$ 100 Δ, is used to obtain a reference solution and parameter (7) is applied to the solutions in D_i. The output point is again at the rear part of the scatterer in order to obtain low-field values to better appreciate slight differences. Non-physical reflections from points A, B, and C are again expected at time steps 160, 230 and 290 respectively.

The first fact to point out is that second- and third-order conditions, although they initially yield better results than first-order ones, cause unstabilities to appear. It should be noted that these unstabilities are due only to the absorbing boundary conditions, since the way in which TLM is formulated ensures its stability [5]. The only second-order condition that produces almost satisfactory results is the Padé approximation of one-way equations. This is in clear contrast with two-dimensional problems in which this type of condition gave the poorest results.

Fig. 5 is a plot of the error parameter versus time for the stable conditions that have been found. Also included is the solution for the second-order space-time extrapolation because it is the best two-dimensional condition. Differences between the reference solution and stable solutions in the interior domain are presented in Fig. 6. No

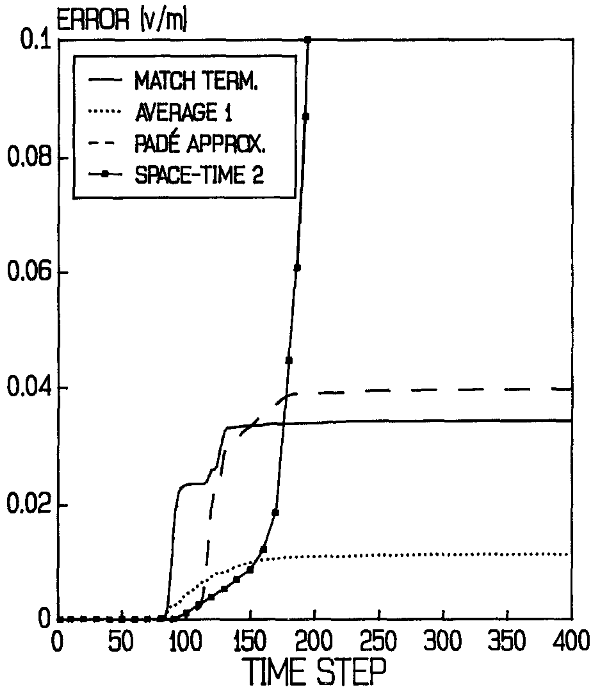

Fig. 5. Comparison of absorbing boundary conditions for the three-dimensional problem.

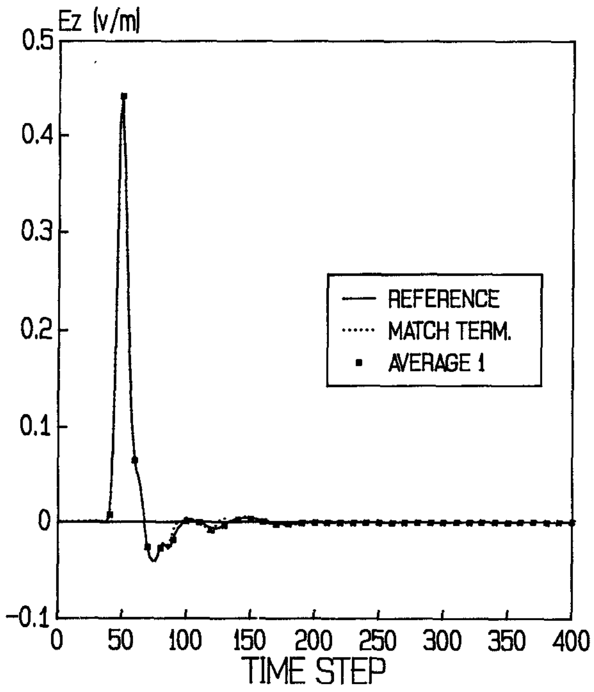

Fig. 6. Electric field at the output point for the reference solution and two absorbing boundary conditions.

stable solution has been found for first-order space-time extrapolation conditions although extrapolation of its second-order behavior in two-dimensional problems seemed to suggest the opposite.

IV. Conclusions

This paper is concerned with the study of absorbing boundary conditions to solve free-space electromagnetic problems using the transmission-line method. Currently, conditions applied to TLM are usually based on reflection coefficients. Two other groups are also included in this paper: one-way conditions for the analytical wave equation and discrete conditions for the discrete wave equation. The need for an explicit study for two and three dimensions has also been established for both analytical and numerical reasons.

For the two-dimensional case, the response of an infinitely-long square cylinder when illuminated by a gaussian pulse has been studied and compared to a reference solution (obtained for a larger domain), by defining an appropriate error parameter. Match termination of the lines are clearly surpassed by discrete boundary conditions. In particular, second-order space-time extrapolation conditions yield the best results without being significantly difficult. One-way equations, although successfully applied for FD-TD calculations, yield stable but very poor results.

Results obtained for two-dimensional cases are not extrapolable to three-dimensional geometries as indicated above. For this case, the response of a cube to a gaussian pulse is studied. Space-time extrapolation techniques do not produce stable conditions in spite of the very good performances exhibited for the two-dimensional case. On the contrary, the best results are those obtained when using first-order averaging conditions. It is a remarkable fact that Padé approximation of one-way equations produces relatively good results in contrast to the poor results obtained for two dimensions.

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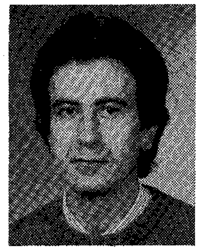

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